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An Equivalence Relation for Torsion-Free Abelian Groups of Finite Rank

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INTRODUCTION

Let G, H be torsion-free abelian groups of finite rank, hereafter simply called "groups." We investigate the equivalence relation obtained by defining $G \approx H$ iff $\{X : \mathrm{Hom}(X, G) = 0\} = \{X : \mathrm{Hom}(X, H) = 0\}$. The relation $\approx$ is the dual to the relation $\equiv$ introduced by Schultz in [S].

We get at the relation $\approx$ by first considering $({}^{\perp}G)^{-}$, the torsion-free class cogenerated by the group G. In Section 1 we give several simple characterizations of the groups in this class.

In Section 2 we consider the classification of those groups G such that $H \approx G$, rank $H =$ rank G implies H is quasi-isomorphic to G. The (somewhat technical) classification is given in Theorem 10.

In Section 3 we define a "core subgroup" for the relation $\approx$. A core subgroup is a pure subgrop G' of G chosen minimal such that $G' \approx G$ or, equivalently, minimal such that $\mathrm{Hom}(K, G') \neq 0$ for all pure subgroups $K \neq 0$ of G. Such a G' is unique up to quasi-isomorphism. Our main result (Theorem 12) is that $G \approx H$ iff the corresponding core subgroups G' and H' are quasi-isomorphic. We also obtain a concrete construction of G' (Theorem 13).

In Section 4 we briefly discuss the analogue of Theorem 12 for Schultz' equivalence relation $\equiv$. This includes the construction of an epimorphic image G^* of G chosen minimal such that $G^* \equiv G$ or, equivalently, such that $\mathrm{Hom}(G^*, W) \neq 0$ for all epimorphic images $W \neq 0$ of G. Such a G^* is unique up to quasi-isomorphism and $G \equiv H$ iff G^* is quasi-isomorphic to H^*.

Notation is standard. We write $H \leqslant G$ ($H < G$) to mean that H is a subgroup (proper subgroup) of G. We use $\lhd$ for a pure subgroup. A subgroup H of a group G is called full if G/H is torsion. A dot stands for the appropriate quasi-concept, e.g., $\dot{\cong}$ and $\doteq$ for quasi-isomorphism and

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quasi-equality. If G is a group we let $\mathrm{E}(G)$ and $\mathrm{QE}(G)$ be the endomorphism and quasi-endomorphism rings of G. The outer type of G is written $\mathrm{OT}(G)$. If S is a set of integral primes and G is a group, let $G_S = Z_S \otimes_Z G$ be the localization of G at S. For G a group and n a non-negative integer we use G^n for the direct sum of n copies of G. The symbol ▮ denotes the end of a proof. We assume familiarity with the standard facts about torsion-free abelian groups of finite rank and quasi-homomorphisms as found in [A] or [F].

1. The Class $({}^{\perp}G)^{\perp}$

Let TF be the class of all (torsion-free abelian finite rank) groups and $G \in TF$. In this section we discuss a class of groups associated with G.

Definition 1. For a group G let ${}^{\perp}G = \{X \in TF : \mathrm{Hom}(X, G) = 0\}$ and $({}^{\perp}G)^{\perp} = \{Y \in TF : \mathrm{Hom}(X, Y) = 0, \forall X \in {}^{\perp}G\}$.

The class $({}^{\perp}G)^{\perp}$ was introduced in [G], where it was shown that $({}^{\perp}G)^{\perp}$ coincides with the reduced groups in an injective class containing the group G.

To state our first theorem, giving some simple equivalent conditions for a group H to be in $({}^{\perp}G)^{\perp}$, we need some notation.

Notation. For G, $H \in TF$ let $H^0[G] = H$, $H^1[G] = \bigcap \{\ker f : f \in \mathrm{Hom}(H, G)\}$ and for $t > 1$, $H^t[G] = (H^{t-1}[G])^1[G]$.

Theorem 2. *Let G and H be groups. The following are equivalent:*

(i) *The group H is in the class $({}^{\perp}G)^{\perp}$.*

(ii) *There exists a descending series of pure subgroups $H = H_0 \rhd H_1 \rhd \cdots \rhd H_n = 0$ such that each H_i/H_{i+1} is isomorphic to a subgroup of G.*

(iii) *For each nonzero pure subgroup K of H, $\mathrm{Hom}(K, G) \neq 0$.*

(iv) *For some integer $t \geqslant 0$, $H^t[G] = 0$.*

Proof. (i) → (ii) Let $H \in ({}^{\perp}G)^{\perp}$. Put $H_0 = H$. Suppose $i \geqslant 0$ and a descending series of pure subgroups $H = H_0 \rhd \cdots \rhd H_i$ has been constructed such that for each $j < i$ the factor group H_j/H_{j+1} is isomorphic to a subgroup of G. If $H_i = 0$ we are done. If $H_i \neq 0$, then the inclusion map is a nonzero element of $\mathrm{Hom}(H_i, H)$ Since $H \in ({}^{\perp}G)^{\perp}$, $H_i \notin {}^{\perp}G$. In this case let $H_{i+1} = \ker g$, where $g: H_i \to G$ is any nonzero map. Note that $H_i \neq 0$ implies that rank $H_{i+1} <$ rank H_i. Thus, we must have $H_n = 0$ for some $n \leqslant$ rank H.

(ii) → (iii) Assume there exists a series of pure subgroups $H = H_0 \rhd H_1 \rhd \cdots \rhd H_n = 0$ with each H_i/H_{i+1} isomorphic to a subgroup of G. Let $0 \neq K$ be pure in H. Let i be the largest nonnegative integer such that K is contained in H_i. Then $i < n$ and $(K + H_{i+1})/H_{i+1}$ is isomorphic to a non-zero subgroup of G. Hence $\mathrm{Hom}(K, G) \neq 0$.

(iii) → (iv) Each $H^t[G]$ is a pure subgroup of G so, by (iii), if $H^t[G] \neq 0$ then $H^{t+1}[G]$ is a proper subgroup of $H^t[G]$. Since H has finite rank we have $H^t[G] = 0$ for some $t \leqslant \mathrm{rank}\, H$.

(iv) → (i) Let $f\colon Y \to H$ for some $Y \in {}^{\perp}G$. Since $Y \in {}^{\perp}G$ it follows by a simple induction argument that image $f \leqslant H^s[G]$ for all positive integers s. By (iv), image $f = 0$ and, hence, $H \in ({}^{\perp}G)^{\perp}$. ∎

Corollary 2.1. *The class $({}^{\perp}G)^{\perp}$ is closed under (finite) sums, extensions, and subgroups. In particular if $H \leqslant G^n$ for some positive integer n then $H \in ({}^{\perp}G)^{\perp}$.*

For a type τ and positive integer r let $r\tau$ be the sum of r copies of τ.

Corollary 2.2. *Let $H \in ({}^{\perp}G)^{\perp}$ with $\mathrm{rank}\, H = r$. Then $\mathrm{OT}(H) \leqslant r\,\mathrm{OT}(G)$.*

Proof. Apply (ii) of Theorem 2 together with the observation that if $0 \to A \to B \to C \to 0$ is exact then $\mathrm{OT}(B) \leqslant \mathrm{OT}(A) + \mathrm{OT}(C)$ [Wa, remark after Corollary 6]. ∎

Corollary 2.3. *Let G be a rank one group of idempotent type τ and H be a group of rank r. Then $H \in ({}^{\perp}G)^{\perp}$ if and only if $H \leqslant G^r$.*

Proof. Only the "only if" part needs proof. By Corollary 2.2, if $H \in ({}^{\perp}G)^{\perp}$ then $\mathrm{OT}(H) \leqslant r\tau$. But if τ is idempotent then $r\tau = \tau$. Hence $\mathrm{OT}(H) \leqslant \tau$ and, consequently, $H \leqslant G^r$. ∎

Remark. Theorem 2 has a straightforward generalization to torsion-free abelian groups of arbitrary rank.

We conclude Section 1 with a further investigation of condition (iv) of Theorem 2. We need a preliminary lemma, which is well known.

Lemma 3. *Let G be a strongly indecomposable group which is not rank one of idempotent type. Then there exists a non-quasi-split extension of G by G.*

Proof. For G as above it is enough to show that $\mathrm{Ext}(G, G) \neq 0$. Then $\mathrm{Ext}(G, G)$ is uncountable and, thus, there exists a non-quasi-split extension of G by G [Wa2, Theorem 4].

Suppose, by way of contradiction, that $\operatorname{Ext}(G, G) = 0$. Then, if $S = \{p : pG \neq G\}$ there exists a full free subgroup F of G such that $(G/F)_S = 0$ [Wi, Theorem 2]. Apply $Z_S \otimes$ to the sequence $0 \to F \to G \to G/F \to 0$ to conclude that $G_S = F_S$. But, by definition of S, $G \cong G_S$. Thus, $G \cong F_S$. Since G is strongly indecomposable the completely decomposable group F_S must be of rank one. Thus, $G \cong Z_S$, a rank one group of idempotent type, contrary to our assumption. ∎

THEOREM 4. *Let G be a group which is not rank one of idempotent type. Suppose that* $\mathrm{QE}(G) = \Gamma$, *Γ a division ring. Then for all $s \geq 0$ there exists a group X such that $X^{s+1}[G] = 0$, $X^s[G] \neq 0$.*

Proof. Let $X(0) = G$. Since $\mathrm{QE}(G) = \Gamma$, $X(0)$ is strongly indecomposable. Thus, by Lemma 3 and our hypothesis, we can let $X(1)$ be a non-quasi-split extension of G by G. Regard $X(0) < X(1)$. Clearly $X(1)[G] \leq X(0)$. Using the fact that $\mathrm{QE}(G) = \Gamma$, i.e., every nonzero endomorphism of G is a quasi-automorphism, it is easy to show that $X(1)[G] = X(0)$.

For $s \geq 2$ we will construct an ascending sequence of groups $X(s)$ satisfying:

$$(\alpha) \quad X(s)/X(s-1) \cong G \qquad \text{and} \qquad (\beta) \quad X(s)^1\,[G] = X(s-1).$$

If $X = X(s)$ conditions (α) and (β) will then imply that $X^{s+1}[G] = 0 \neq X^s[G]$, as desired.

The construction is by induction. Let $s \geq 2$ and suppose $X(t)$ has been constructed for all $t < s$. Let W be a non-quasi-split extension of $X(s-1)/X(s-2)$ $(\cong G)$ by G and construct $X(s) > X(s-1)$ such that $X(s)/X(s-2) \cong W$ [F, Proposition 24.6].

We claim that $X(s)$ satisfies conditions (α) and (β). Condition (α) follows immediately from the definitions of W and $X(s)$. Moreover, by condition(α), $X(s)^1\,[G] \leq X(s-1)$.

To verify (β), suppose by way of contradiction that $X(s)^1\,[G]$ is a proper subgroup of $X(s-1)$. Then there exists $f\colon X(s) \to G$ with $f[X(s-1)] \neq 0$. Since, by induction, $X(s-1)^1\,[G] = X(s-2)$, the map f induces a map $f'\colon X(s)/X(s-2) \to G$. Let f'' be the restriction of f' to $X(s-1)/X(s-2)$. Again by induction $X(s-1)/X(s-2) \cong G$ and by assumption $\mathrm{QE}(G) = \Gamma$. Therefore, since $f'' \neq 0$, f'' must be a quasi-automorphism. It follows that $X(s-1)/X(s-2)$ is a quasi-summand of $X(s)/X(s-2)$, contradicting the construction of $X(s)$. ∎

2. The Relation $\approx$

We now define the relation that is the main focus of the paper.

Definiton 5. Let G and H be groups. Define $G \approx H$ if $({}^{\perp}G)^{\perp} = ({}^{\perp}H)^{\perp}$.

We list for reference some easily verified properties of the relation $\approx$.

Proposition 6. (a) *The relation* $\approx$ *is an equivalence relation.*

(b) $G \approx H$ *if and only if* $H \in ({}^{\perp}G)^{\perp}$ *and* $G \in ({}^{\perp}H)^{\perp}$.

(c) *If* $G \doteq H$ *then* $G \approx H$.

(d) $G \approx H$ *if and only if* ${}^{\perp}G = {}^{\perp}H$.

(e) $G \approx H$ *if and only if for every nonzero pure subgroup* $K \leqslant G$ *we have* $\mathrm{Hom}(K, H) \neq 0$ *and for every nonzero pure subgroup* $L \leqslant H$ *we have* $\mathrm{Hom}(L, G) \neq 0$.

Proof. Parts (a)–(c) are immediate from the definitions. Part (d) follows from the fact that ${}^{\perp}[({}^{\perp}G)^{\perp}] = {}^{\perp}G$ for any group G. Part (e) follows from part (b) and Theorem 2(iii). ∎

Remarks. (1) It follows from part (d) that our relation is dual to the one introduced by Schultz in [S].

(2) Another way to view the relation $\approx$ is that $G \approx H$ if and only if G and H cogenerate the same torsion-free class in the category of torsion-free finite rank groups and (quasi) homomorphisms.

Theorem 2 tells us that $G \approx H$ if and only if G and H are built up in "layers," each constituent layer of one group isomorphic to a subgroup of the other. Thus, we do not expect G and H necessarily to look substantially alike if $G \approx H$. In particular, if H is any extension of G by G then $H \approx G$. Of course, this kind of example is eliminated if we also require that $\operatorname{rank} H = \operatorname{rank} G$. For the remainder of Section 2 we investigate the following problem.

Problem. *Characterize those groups* G *such that, for all groups* H, *if* $\operatorname{rank} H = \operatorname{rank} G$ *and* $H \approx G$ *then* $H \doteq G$.

For our next theorem we need some additional concepts.

If $A \leqslant B$ let A_* be the pure subgroup of B generated by A. Call A quasi-pure in B if A_*/A is finite. Call an embedding $f\colon G \to H$ a quasi-pure embedding if $f(G)$ is quasi-pure in H.

Theorem 7. *Let* G *and* H *be groups with* $H \in ({}^{\perp}G)^{\perp}$. *Suppose that*

QE(G) *is a division ring. Then any nonzero element of* Hom(G, H) *is a quasi-pure embedding of G into H.*

Proof. Let $H \in ({}^{\perp}G)^{\perp}$ and let $H = H_0 \rhd H_1 \rhd \cdots \rhd H_n = 0$ and $f_i: H_i/H_{i+1} \to G$ be the pure subgroups and embeddings given in Theorem 2(ii). Suppose $0 \neq \phi: G \to H$. Let $j \leqslant n-1$ be the largest index such that $\phi(G) \leqslant H_j$. Then, if $\pi: H_j \to H_j/H_{j+1}$ is the natural map, we have $f_j\pi\phi \neq 0$ in E(G). Since QE(G) is a division ring there exists $\delta \in$ E(G) and a positive integer t with $\delta f_j \pi \phi = t1_G$. It follows that ϕ is monic and that $H_j \doteq \phi(G) \oplus \ker(\delta f_j \pi) = \phi(G) \oplus H_{j+1}$. Since H_j is pure in H and since quasi-purity is transitive, $\phi(G)$ is quasi-pure in H ▮

Corollary 7.0. *Let G and H be groups with $G \approx H$ and with* QE(G) *a division ring. Then there exists a quasi-pure embedding from G into H.*

Proof. By Proposition 6, if $H \approx G$ then $H \in ({}^{\perp}G)^{\perp}$ and ${}^{\perp}H = {}^{\perp}G$. Since Hom$(G, G) \neq 0$ the latter condition implies that Hom$(G, H) \neq 0$ as well. Thus we can apply Theorem 7 to obtain the desired quasi-embedding. ▮

Corollary 7.1. *Let A be a rank one group of idempotent type and B be a group. Then $B \approx A$ if and only if $B \doteq A \oplus K$ with* OT$(K) \leqslant$ type A.

Proof. Let A be rank one of idempotent type and B be a group with $B \approx A$. Corollary 7.0 gives us a quasi-pure embedding of A into B. But OT$(B) \leqslant$ type A by Corollary 2.3. It follows that $B \doteq A \oplus K$ for some K by Proposition 1.7 of [VW]. Plainly, OT$(K) \leqslant$ type A. The converse follows from Proposition 6(e). ▮

Corollary 7.2. *Let G and H be groups with $G \approx H$ and with* QE(G) *a division ring. Then if either* rank $H =$ rank G *or* QE(H) *is a division ring we have $G \doteq H$.*

Proof. Let G, H be groups with $G \approx H$ and with QE(G) a division ring. By Corollary 7.0 we have a quasi-pure embedding of G into H. Thus if rank $H =$ rank G then $G \doteq H$. If we instead assume that QE(H) is also a division ring then we have quasi-pure embeddings in both directions, so rank $H =$ rank G. ▮

The following corollary may be of some independent interest.

Corollary 7.3. *Let G and H be groups with* QE(G) *a division ring such that $G \leqslant H^n$ and $H \leqslant G^n$ for some positive integer n. Then if* QE(H) *is a division ring or if* rank $H =$ rank G, *G and H are quasi-isomorphic.*

Proof. If $G \leqslant H^n$ and $H \leqslant G^n$ then $G \approx H$. ▮

We need one more lemma before giving the main results of this section. For convenience we introduce an additional bit of notation.

Notation. For a group G with nonzero nilpotent endomorphisms let K_λ be the kernel of any $\lambda \in E(G)$ such that $0 = \lambda^2 \neq \lambda$.

LEMMA 8. (a) *The set K_λ is a nonzero proper pure subgroup of G.*

(b) *The map $x + K_\lambda \to \lambda x$ is an embedding of G/K_λ into K_λ.*

(c) $G \approx K_\lambda$.

Proof. Parts (a) and (b) are immediate. Since $K_\lambda \leqslant G$ plainly ${}^\perp G \leqslant {}^\perp K_\lambda$. Thus, to prove (c) it suffices to show that ${}^\perp K_\lambda \leqslant {}^\perp G$ or, equivalently, that $\mathrm{Hom}(X, G) \neq 0$ implies $\mathrm{Hom}(X, K_\lambda) \neq 0$. Let X be a group with $0 \neq f\colon X \to G$. Then $W = \operatorname{image} f$ is a nonzero subgroup of G and to prove that $\mathrm{Hom}(X, K_\lambda) \neq 0$ it is enough to show that $\mathrm{Hom}(W, K_\lambda) \neq 0$. This clearly holds if $W \leqslant K_\lambda$. If W is not contained in K_λ we can use part (b) to conclude $\mathrm{Hom}(W, K_\lambda) \neq 0$. ▌

For the remainder of this section we let $\mathrm{N}[R]$ be the nil radical of a ring R.

THEOREM 9. *Let G be a strongly indecomposable group. Then the following are equivalent*:

(i) *If H is a group with* rank H = rank G *and $H \approx G$ then $H \doteq G$.*

(ii) $\mathrm{QE}(G)$ *is a division ring.*

Proof. First note that if G is strongly indecomposable then $\mathrm{QE}(G)$ is an Artinian algebra with no proper idempotents. Thus $\mathrm{QE}(G)$ is a division ring if and only if $\mathrm{N}[\mathrm{QE}(G)] = 0$. It is easy to check that $\mathrm{N}[\mathrm{QE}(G)] = 0$ if and only if $\mathrm{N}[\mathrm{E}(G)] = 0$.

We prove (i) → (ii) by contradiction. Let G be strongly indecomposable with $\mathrm{QE}(G)$ not a division ring. Then $\mathrm{N}[\mathrm{E}(G)] \neq 0$ so there exists $\lambda \in \mathrm{E}(G)$ with $0 = \lambda^2 \neq \lambda$. Let K_λ be as previously defined and set $H = K_\lambda \oplus G/K_\lambda$. Plainly, rank H = rank G and H and G are not quasi-isomorphic. But, using Lemma 8, it is easy to check that $H \approx K_\lambda \approx G$. Thus we have contradicted condition (i).

The converse implication, (ii) → (i), follows from Corollary 7.2. ▌

EXAMPLE. Let A be a rank one group of type $[(1, 1, 1, \ldots)]$ and let G be such that $0 \to A \to G \to A \to 0$ is exact. By Corollary 2.1, $A \in ({}^\perp G)^\perp$ and by Theorem 2(ii), $G \in ({}^\perp A)^\perp$. Thus, by Proposition 6(ii), $A \approx G$. It is not difficult to choose rank two groups G_1 and G_2, each an extension of A by A, such that G_1 and G_2 are strongly indecomposable and non quasi-isomorphic. But, for any such G_1, G_2 we have $G_1 \approx G_2 \approx A$.

Remarks. (1) For any group G as in the example $\mathrm{QE}(G)$ is isomorphic to the ring of rational matrices of the form $\left[\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right]$. Thus, $\mathrm{QE}(G)$ is not a division ring.

(2) As noted in [G], the groups G and A also provide an example of the situation $G \in ({}^{\perp}A)^{\perp}$ but G not embeddable in A^n. This is in contrast to Corollary 2.3.

We conclude this section with a theorem providing a solution to our previously posed problem.

THEOREM 10. *For a group G the following are equivalent*:

(i) *If H is a group with* rank $H = \operatorname{rank} G$ *and* $H \approx G$ *then* $H \doteq G$.

(ii) $G \doteq Z^s \oplus (\bigoplus_{i=1}^{r} G_i)$, *where* $r, s \geqslant 0$, *and the G_i's are non-free groups satisfying* (a) $\operatorname{Hom}(G_i, G_j) = 0$ *for* $i \neq j$, (b) *each* $\mathrm{QE}(G_i) = \Gamma_i$, Γ_i *a division ring, and* (c) *each subgroup of rank* $\leqslant s$ *of any* G_i *is free.*

Proof. Suppose G satisfies condition (i). Since G is torsion-free of finite rank there is a quasi-decomposition $G \doteq Z^s \oplus G_1 \oplus \cdots \oplus G_r$, where $r, s \geqslant 0$ and each G_i is non-free and strongly indecomposable. We will verify parts (a), (b), and (c) of condition (ii).

Say $0 \neq f\colon G_i \to G_j$ for $i \neq j$. Let $K = \ker f$. If $K \neq 0$ let $H = Z^s \oplus K \oplus (G_i/K) \oplus (\bigoplus_{t \neq i} G_t)$. Plainly, rank $H = \operatorname{rank} G$ and G and H are not quasi-isomorphic. But, since G_i/K embeds in G_j, it is easy to verify the statement of Proposition 6(e). It follows by Proposition 6 that $H \approx G$—contradicting condition (i). Thus $K = 0$ and G_i embeds in G_j. But then we can take $H = F \oplus (\bigoplus_{t \neq i} G_t)$, where F is free of rank $= s + \operatorname{rank} G_i$, to obtain a similar contradiction of condition (i). Hence, $\operatorname{Hom}(G_i, G_j) = 0$ for $i \neq j$.

Next, let $N_i = \mathrm{N}[\mathrm{E}(G_i)]$ and suppose some $N_i \neq 0$. Let $0 \neq \lambda \in E(G_i)$ with $\lambda^2 = 0$ and modify the quasi-decomposition for G, replacing the quasi-summand G_i by $K_\lambda \oplus G_i/K_\lambda$. If H is the modified quasi-decomposition then, again, it is easy to check that rank $H = \operatorname{rank} G$ and $H \approx G$ but H is not quasi-isomorphic to G. Thus, each $N_i = 0$ and, as in Theorem 9, each $\mathrm{QE}(G_i) = \Gamma_i$.

Finally, if some G_i contains a non-free subgroup W of rank $u \leqslant s$ then we can employ the group $H = Z^{s-u} \oplus W \oplus (\bigoplus_t G_t)$ to contradict condition (i). This completes the proof that (i) $\to$ (ii).

Conversely, suppose G satisfies condition (ii) and let H be such that rank $H = \operatorname{rank} G$ and $H \approx G$. Since $H \in ({}^{\perp}G)^{\perp}$ there exists a series $H = H_0 \rhd H_1 \rhd \cdots \rhd H_n = 0$ of pure subgroups and embeddings $\alpha_i\colon H_i/H_{i+1} \to G$. These embeddings plainly can be chosen such that $\alpha_i(H_i/H_{i+1}) \leqslant$ some copy of Z or $\alpha_i(H_i/H_{i+1}) \leqslant G_t$ for some integer $t = t(i)$ with $1 \leqslant t \leqslant r$.

For each fixed j, $1 \leqslant j \leqslant r$, there exists a nonzero map $\theta = \theta(j)\colon G_j \to H$. Let $i = i(j)$ be the largest index such that $\theta(G_j) \leqslant H_i$. Let $\pi\colon H_i \to H_i/H_{i+1}$

be the natural map. Then $\alpha_i\pi\theta$ is a nonzero map in $\mathrm{Hom}(G_j, Z)$ or in $\mathrm{Hom}(G_j, G_t)$, where $t = t(i(j))$. Since $\mathrm{Hom}(G_j, Z) = 0$ and $\mathrm{Hom}(G_j, G_t) = 0$ for $t \neq j$ we must have the second case with $t = j$. That is, $\alpha_i\pi\theta$ is a nonzero map in $\mathrm{QE}(G_j)$. Using the fact that $\mathrm{QE}(G_j)$ is a division ring and arguing as in Theorem 7 we obtain that θ is monic and that $H_i \doteq \theta(G_j) \oplus H_{i+1}$. Thus, for $1 \leqslant j \leqslant r$, each G_j is isomorphic to a quasi-summand of some $H_{i(j)}$ with complementary summand $H_{i(j)+1}$.

Let $I = \{i(1), ..., i(r)\}$ and suppose $i \notin I$. If α_i maps H_i/H_{i+1} into a copy of Z or if rank image $\alpha_i \leqslant s$ then $H_i = H_{i+1} \oplus F_i$ with F_i free. The remaining possibility, image $\alpha_i \leqslant$ some G_t, rank image $\alpha_i > s$, cannot occur since rank $H = \sum$ rank $H_i/H_{i+1} = \sum_{i \in I}$ rank $G_i + \Sigma_{i \notin I}$ rank $H_iH_{i+1} =$ rank G. It follows that $\Sigma_{i \notin I}$ rank $H_i/H_{i+1} = s$. Thus, $i \notin I$ implies rank image $\alpha_i \leqslant s$. Therefore, for all $i \notin I$, we have $H_i = H_{i+1} \oplus F_i$ with F_i free.

The preceding two paragraphs together imply, working up through the chain $H = H_0 \rhd H_1 \rhd \cdots \rhd H_n = 0$, that H is quasi-isomorphic to G. ▮

3. Core Subgroups

Definition 11. For a group G, a core subgroup is a pure subgroup G' of G chosen minimal such that $G' \approx G$.

Plainly, for any G of finite rank, we can choose such a G'. Any such G' will be a minimal pure subgroup of G which is a cogenerator for the set of all pure subgroups of G.

We come to our main result, giving the connection between our relation $\approx$ and quasi-isomorphism.

Theorem 12. (a) *A core subgroup G' of a group G is unique up to quasi-isomorphism.*

(b) *Let G', H' be core subgroups of groups G, H. Then $G \approx H$ if and only if $G' \doteq H'$.*

Proof. Let G', H' be core subgroups of groups G, H. Since $\approx$ is an equivalence relation $G \approx H$ if and only if $G' \approx H'$.

We claim that any core subgroup will satisfy condition (ii) of Theorem 10. To prove the claim let G' be a core subgroup of G and suppose that $0 = \lambda^2 \neq \lambda \in E(G')$. Then $K_\lambda = \ker \lambda$ is a proper pure subgroup of G', hence pure in G. Moreover, by Lemma 8, $K_\lambda \approx G' \approx G$, contradicting the minimality of G'. It follows that $\mathrm{QE}(G')$ has no nilpotent elements and, since $\mathrm{QE}(G')$ is Artinian, it must be a ring direct product of division rings. Say $\mathrm{QE}(G') = \Gamma_1 \times \cdots \times \Gamma_r$ with each Γ_i a division ring. Then, by a standard argument, $G' \doteq G_1 \oplus \cdots \oplus G_r$, where the G_i's satisfy $QE(G_i) = \Gamma_i$ and $\mathrm{Hom}(G_i, G_j) = 0$ for $i \neq j$. Thus G' satisfies Theorem 10(ii) with $s = 0$.

Now let G', G'' be core subgroups of G. Then $G' \approx G'' \approx G$ and both G' and G'' satisfy condition (ii) of Theorem 10. The argument (ii) → (i) of Theorem 10 can be employed to obtain quasi-pure embeddings of G' into G'' and vice versa. Hence $\operatorname{rank} G' = \operatorname{rank} G''$. We can now apply Theorem 10 to conclude that $G' \doteqdot G''$. This proves part (a) of Theorem 12.

More generally, a trivial modification of the argument of the preceding paragraph shows that if G', H' are core subgroups of G, H with $G' \approx H'$ then $G' \doteqdot H'$. Since $G \approx H$ is and only if $G' \approx H'$ part (b) of Theorem 12 holds as well. ▮

COROLLARY 12.1. *Let G' be a core subgroup of G. Then* $\mathrm{QE}(G')$ *is a product of division rings and if $G' < K < G$ then* $\mathrm{QE}(K)$ *is not a product of division rings.*

Proof. In the proof of Theorem 12 we established the fact that $\mathrm{QE}(G')$ is a product of division rings. Furthermore, the proof of Theorem 10 shows that if K is a group with $\mathrm{QE}(K)$ a product of division rings and H is a group with $H \approx K$ then there is a quasi-pure embedding of K into H.

It is easy to see that if K is a group with $G' \leqslant K \leqslant G$ then $G' \approx K \approx G$. But, since G' and K are pure subgroups of G with $G' < K$ then $\operatorname{rank} G' < \operatorname{rank} K$. Hence, plainly, there is no embedding of K into G'. Thus, if $G' < K < G$, $\mathrm{GE}(K)$ cannot be a product of division rings. ▮

Remark. The property above is not a characterizing property for a core subgroup. Let $G = A \oplus Z$, where A is rank one of type $[\langle \infty, 0, 0, \ldots \rangle]$. Then $G' = A$ but both A and Z satisfy the property of Corollary 12.1.

For a group G, we can give an algorithm to construct a core subgroup G'. Let G be a group of rank r. Define, for $1 \leqslant i \leqslant r$ pure subgroups $G(i)$ of G as follows: $G(1) = G$ and if $G(i-1)$ has been defined for $1 < i \leqslant r-1$ define $G(i)$ in one of the two following ways:

Case 1. If $\mathrm{E}[G(i-1)]$ has no nonzero nilpotent elements let $G(i) = G(i-1)$. Note that in this case we will put $G(j) = G(i-1)$ for $i-1 \leqslant j \leqslant r$.

Case 2. If $\mathrm{E}[G(i-1)]$ has nonzero nilpotent elements let $G(i) = K_\lambda$ for some $0 = \lambda^2 \neq \lambda \in \mathrm{E}[G(i-1)]$. Note that in this case $\operatorname{rank} G(i) < \operatorname{rank} G(i-1)$.

THEOREM 13. *For a group G with notation as above $G(r)$ is a core subgroup of G.*

Proof. For all $1 < i \leqslant r$ each $G(i) \approx G(i-1)$ and, hence, $G(r) \approx G$. Thus, to show that $G(r)$ is a core subgroup we need to show that $G(r)$ is not equivalent to any proper pure subgroup of $G(r)$. To see this we first claim that $E[G(r)]$ has no nonzero nilpotent elements. The proof of the claim is by contradiction. Suppose otherwise. Then, by the note of Case 1, each $E[G(i)]$ must have had nilpotent elements for $1 \leqslant i \leqslant r$. Thus, by the note of Case 2, we have rank $G(i+1) <$ rank $G(i)$ for $1 \leqslant i < r$. It follows that rank $G(r) = 1$, contradicting the assumption that $E[G(r)]$ had nonzero nilpotent elements. This proves the claim. As before $QE[G(r)]$ is a product of division rings and the argument (ii) → (i) of Theorem 10 shows that $G(r)$ cannot be equivalent to any proper pure subgroup of $G(r)$. Thus, by definition, $G(r)$ is a core subgroup of G. ∎

4. The Relation ≡

In this section we briefly turn our attention to the relation introduced by Schultz in [S].

Definition 14. [S]. Let G and H be groups. Define $G \equiv H$ if $G^{\perp} = H^{\perp}$.

Let G be a group of rank r. By considering a series of successive factor groups $G = G[1], G[2], ..., G[r]$ of G with $G[i+1] = G[i]$ if $E(G[i])$ has no nonzero nilpotent elements, $G[i+1] = G[i]/(\text{image } \lambda)_*$, if $0 = \lambda^2 \neq \lambda \in E(G[i])$, we obtain a "core epimorphic image" $G[r]$ of G. Any of these $G[r]$'s (corresponding to different choices of the λ's) will be quasi-isomorphic. A core epimorphic image of G plays the role for the relation $\equiv$ analogous to that played by a core subgroup of G for the relation $\approx$. More precisely, we have the following theorem.

Theorem 15. (a) *Let G be a group and let $G^{\#}$ be a core epimorphic image of G. Then $G \equiv G^{\#}$ and $G \not\equiv K$ for any proper epimorphic image K of $G^{\#}$.*

(b) *Let G, H be groups with core epimorphic images $G^{\#}$, $H^{\#}$. Then $G \equiv H$ if and only if $G^{\#} \doteq\cong H^{\#}$.*

The proof of Theorem 15 is by a sequence of arguments dual to those already presented.

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